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Analogue Simulation of Bending and Buckling of Nonlinear Elastic Bars

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SUMMARY

A computing method is discussed for the problem of bending and buckling of nonlinear elastic bars. Arbitrary stress-strain relations are allowed. Analogue simulation proved to be adequate. Combining of the computing circuits for the basic parts of the computation is realized by nesting in different time-scales.

1. Introduction

The usual practice in the theory of calculating the bending and buckling of loaded bars is to assume the relation between stress (σ) and strain (ε) to be linear according to Hooke's law $\sigma = E\varepsilon$, where E is the material's modulus of elasticity.

The assumption of linearity is essential. It yields an explicit algebraic relation between the bending moment in a certain cross-section and the curvature of the bar at that place making it possible to find an analytical solution for the problems of both bending and buckling. In case of nonlinear elasticity two integral relations appear for the bending moment M and the compressive force N respectively, each relation having as implicit unknowns the curvature of the bar and the stress of the axis of the bar. An iterative method is used to find the unknowns out of these integral relations.

To solve the whole problem of bending and buckling of a nonlinear elastic bar analogue simulation proved to be adequate. The described procedure is useful for arbitrary (but elastic) stress-strain relations.

Mathematical Formulation

Consider an infinitesimal element dx (see Figure 1) of a pinned bar. Forces (N, D, Q) and moments (M) act as indicated. Besides there are continuously distributed loads with intensities n (horizontal) and q (vertical), and also an elastic lateral load $-cw(x)$, where $w(x)$ is the lateral deflection. Furthermore a local moment (M_1) and a local force (Q_2), acting on the bar at arbitrary x_1 and x_2 respectively, are indicated.

Under static load conditions equilibrium should exist for the horizontal forces, the vertical forces and the moments. Over the infinitesimal element dx these conditions yield three differential equations, respectively

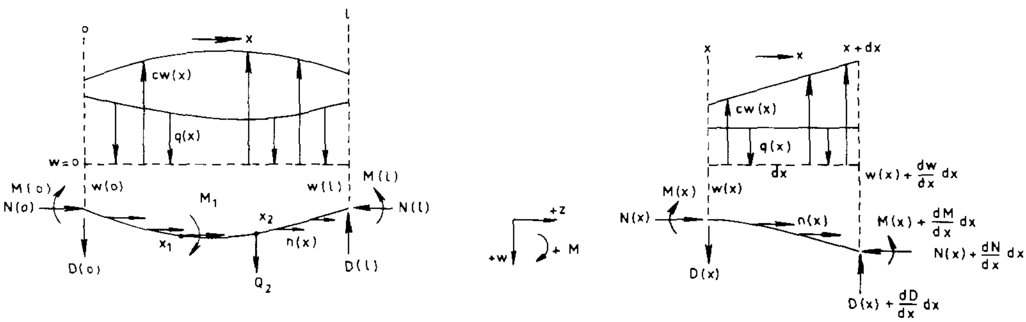


Figure 1. Loading of a pinned bar and equilibrium with respect to x.

$$\frac{dN}{dx} - n = 0 \tag{1}$$

$$\frac{dD}{dx} - q + cw(x) - \delta(x - x_2)Q_2 = 0 \tag{2}$$

$$\frac{dM}{dx} - N(x)\frac{dw}{dx} + D_x - \delta(x - x_1)M_1 = 0 . \tag{3}$$

The force Q_2 and the moment M_1 are introduced in these equations by means of a δ-function (defined by $\delta(x) \equiv 0$ for $x \neq 0$ and $\int_{-\infty}^{+\infty} \delta(x)dx = 1$). It will be allowed that the deflection and/or the angular rotation of the bar is prescribed for certain values of x. Such conditions complicate the problem because they result into implicitly given values of the reaction forces Q and the reaction moments M for these values of x.

For small angles of rotation $\theta = dw/dx$ there is a simple relation between the curvature $1/R(x)$ and the second derivative of the deflection $w(x)$:

$$\frac{d^2 w}{dx^2} = - \frac{1}{R(x)} . \tag{4}$$

To find a unique solution of the differential equations (1), (2), (3), (4) five boundary values are needed. Mostly they are specified at both ends of the bar, for instance for a pinned bar the boundary values are $w(0)$, $N(0)$, $M(0)$, $w(l)$ and $M(l)$. So the considered problem belongs to the class of two-point boundary-value problems.

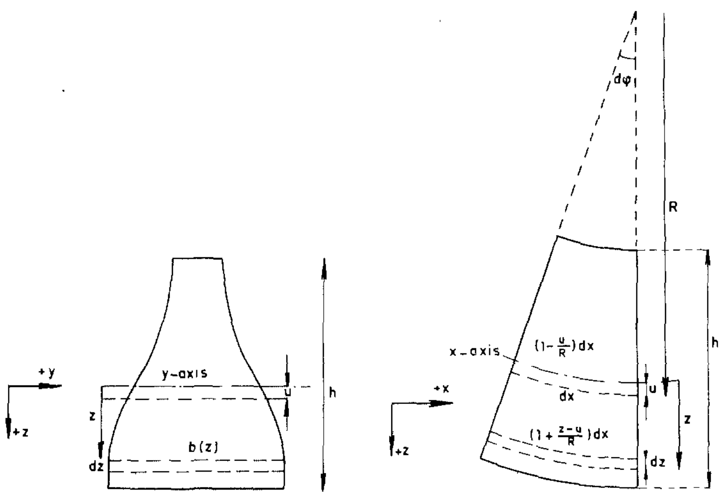


Figure 2. Equilibrium in a cross-section and ditto with respect to x dependent on each other via the curvature.

To be able to solve the set of differential equations (1), (2), (3) and (4) finally it is necessary to describe the relation between the curvature $1/R$ and N, D and M for each x. A consideration of stresses and strains in a cross-section yields (see Figure 2) (the influence of shearing is neglected):

$$N = - \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} b(z)\sigma\left(\frac{z-u}{R}\right) dz \tag{5}$$

$$M = \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} b(z)z\sigma\left(\frac{z-u}{R}\right) dz . \tag{6}$$

Figure 2 shows also that the relative stress $\varepsilon(z)$ equals $(z-u)/R$, where u is the deflection of the neutral layer.

From the general formulas (5) and (6) the simple algebraic relations for a rectangular cross-section and a linear stress-strain relation are derived easily:

$$N = b\,h\,E\frac{u}{R}, \tag{7}$$

$$M = \frac{EI}{R}, \tag{8}$$

where EI is the flexural rigidity of the bar. It appears that the curvature $1/R$ is only dependent on the moment M, while the relative stress of the bar's axis u/R (see figure 2) is only dependent on the compressive force N. For more details about theoretical aspects of the considered problem we refer to handbooks, e.g. [1].

The block diagram of figure 3a shows very schematically the way of solving the differential equations (1), (2), (3), (4) and the integral relations (5) and (6). For simplicity only one local moment M_1, one local force Q_2, one prescribed rotation θ_3 and one prescribed deflection w_4 are indicated. Special attention will be paid to the block representing the iteration process to solve relations (5) and (6) and to the block which determines the implicitly given reaction forces and moments. Only a brief explanation will be given of the level-control blocks for solving the two-point boundary-value aspect of the problem.

2. The Iteration-Process over a Cross-Section

In the general case of a nonlinear stress-strain relation a method should be found to solve the unknown curvature $1/R$ out of the relations (5) and (6) for any given pair of N and M. An iteration-process is the obvious means for this purpose. The linear formulas (7) and (8) give an indication in which way a suitable iteration-process can be designed. We will show below that the convergence of this iteration-process is equivalent with the stability of the equilibrium (5), (6) of the loading in a cross-section. (In [2] it was already proved that $(d\sigma/d\varepsilon) \geqq 0$ is a sufficient condition for convergence).

Denoting the approximation of u/R and $1/R$ obtained in the n-th iteration cycle by $\{u/R\}_n$ and $\{1/R\}_n$, then substitution of $\{u/R\}_n$ and $\{1/R\}_n$ in (5) and (6) yields values N_n and M_n according to

$$N_n = -\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} b(z)\sigma\left(\left\{\frac{1}{R}\right\}_n z - \left\{\frac{u}{R}\right\}_n\right)dz \tag{9}$$

$$M_n = \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} b(z)z\sigma\left(\left\{\frac{1}{R}\right\}_n z - \left\{\frac{u}{R}\right\}_n\right)dz, \tag{10}$$

which will generally differ from the prescribed N and M. These differences $N-N_n$ and $M-M_n$ are used now to find the next approximation $\{u/R\}_{n+1}$ and $\{1/R\}_{n+1}$ by using the iteration formulas:

$$\left\{\frac{u}{R}\right\}_{n+1} = \left\{\frac{u}{R}\right\}_n + K_N(N-N_n) \tag{11}$$

$$\left\{\frac{1}{R}\right\}_{n+1} = \left\{\frac{1}{R}\right\}_n + K_M(M-M_n), \tag{12}$$

where K_N and K_M are two positive constants.

To prove the convergence of the iteration-process, we introduce the error vector $\boldsymbol{\varepsilon}_n = \begin{pmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \end{pmatrix}$, where

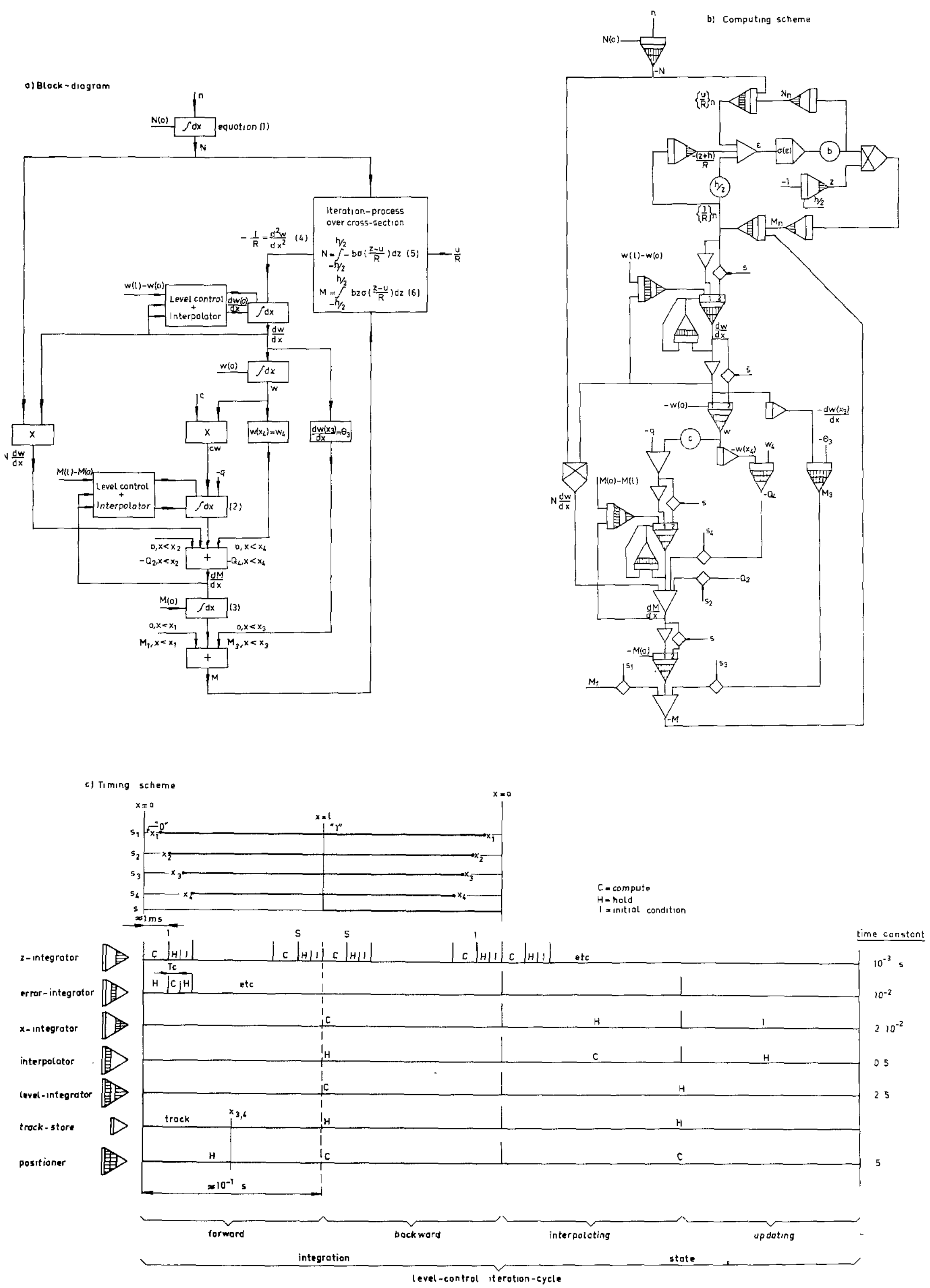


Figure 3. Block-diagram of the computation, analogue computing scheme and timing scheme for a pinned bar.

$$\varepsilon_{1n} = \left\{\frac{u}{R}\right\}_n - \frac{u}{R}, \tag{13}$$

$$\varepsilon_{2n} = \left\{\frac{1}{R}\right\}_n - \frac{1}{R}. \tag{14}$$

For small $\|\varepsilon_n\|$ the iteration formulas (11) and (12) can approximately be written as

$$\varepsilon_{n+1} = \begin{pmatrix} 1 - K_N \dfrac{\partial N}{\partial(u/R)} & -K_N \dfrac{\partial N}{\partial(1/R)} \\ -K_M \dfrac{\partial M}{\partial(u/R)} & 1 - K_M \dfrac{\partial M}{\partial(1/R)} \end{pmatrix} \varepsilon_n \tag{15}$$

For a stable equilibrium of a physical element of the bar it is necessary that

i) for constant $1/R$ an extra force dN should do positive work, i.e. the corresponding relative displacement $d(u/R)$ should have the same direction as dN,

ii) for constant u/R an extra moment dM should do positive work, i.e. the corresponding angular rotation $d(1/R)$ should have the same direction as dM,

iii) for constant M an extra force dN should do positive work.

Considering $N = N(u/R, 1/R)$ and $M = M(u/R, 1/R)$ from these conditions it follows

$$\frac{\partial N}{\partial \frac{u}{R}} > 0, \quad \frac{\partial M}{\partial \frac{1}{R}} > 0, \quad \left(\frac{\partial M}{\partial \frac{1}{R}} \frac{\partial N}{\partial \frac{u}{R}} - \frac{\partial N}{\partial \frac{1}{R}} \frac{\partial M}{\partial \frac{u}{R}} \right) > 0. \tag{16}$$

Convergence of the iteration-process is guaranteed if the eigenvalues of the transformation-matrix T in (15) satisfy $|\lambda_{1,2}| < 1$.

The eigenvalues $\lambda_{1,2}$ are roots of the equation

$$P(\lambda) = \lambda^2 - \lambda \left(2 - K_N \frac{\partial N}{\partial \frac{u}{R}} - K_M \frac{\partial M}{\partial \frac{1}{R}} \right) + |T| = 0 \tag{17}$$

where $|T|$ means the determinant of matrix T. The condition $|\lambda_{1,2}| < 1$ is only satisfied if

$$|T| \simeq 1 - K_N \frac{\partial N}{\partial \frac{u}{R}} - K_M \frac{\partial M}{\partial \frac{1}{R}} < 1 \tag{18}$$

$$P(1) = K_N K_M \left(\frac{\partial N}{\partial \frac{u}{R}} \frac{\partial M}{\partial \frac{1}{R}} - \frac{\partial N}{\partial \frac{1}{R}} \frac{\partial M}{\partial \frac{u}{R}} \right) > 0, \tag{19}$$

$$P(-1) \simeq 4 - 2K_N \frac{\partial N}{\partial \frac{u}{R}} - 2K_M \frac{\partial M}{\partial \frac{1}{R}} > 0, \tag{20}$$

where the approximations in (18) and (20) relate to terms with products of the small K's. Considering the unequalities (16) obviously conditions (18) and (19) are satisfied and (20) can always be satisfied by choosing K_N and K_M small enough.

The conclusion is that for sufficiently small values of K_N and K_M the iteration process is convergent as long as the physical conditions (16) are satisfied. Apparently the convergence of the iteration-process is equivalent with the stability of the physical system.

To get an idea about practical values of K_N and K_M consider again the linear case (7), (8). Optimal values of K_N and K_M can be calculated now:

$$K_N = \frac{1}{bhE}, \quad K_M = \frac{1}{EI}, \tag{21}$$

for which the first iteration cycle yields the correct values of u/R and $1/R$ independent of the initial guesses.

3. The Level-Control Method

Since the boundary conditions for the equations (1), (2), (3), (4) are normally specified at both ends of the bar the computing problem belongs to the class of two-point boundary-value problems. The level-control method is applied to realize that the boundary values are satisfied. Referring to [3] for a comprehensive description of the level-control method, here only the basic principles of the method are presented:

i) The two-point boundary-value problem is transformed into an unconditionally stable initial value problem by integrating periodically over equal time-intervals forward, i.e. from beginning to end, and backward, i.e. from end to beginning of the bar. For this initial-value problem the remaining boundary conditions are interpreted as level-value conditions;

ii) Level-controls are introduced in order to realize that these level-value conditions are satisfied in the final stationary state;

iii) To obtain asymptotic stability after each period the state of the system is updated where as estimates for the unknown initial values weighted interpolations are taken between the previous estimates and the values at the end of the period.

The method is an iterative one, having a large convergence region though the convergence itself is slow. The method is very suitable for the study of buckling because it remains convergent while passing the buckling condition.

4. Dynamic Positioning of a Simulated Bar

The previous paragraphs dealt with load conditions including local moments and forces. However, locally prescribed angles of rotation and deflections were not included. These result into unknown, implicit reaction moments and reaction forces respectively.

A method was applied to realize this positioning by changing the original time-independent problem into a dynamic one such that the positioning in the final state is the correct one.

Say for simplicity only the displacement in two points is prescribed; the angular rotation in one point, the deflection in the other one. Denoting the difference between the present and the prescribed displacement vector as $\boldsymbol{v}$, an unknown loading $\boldsymbol{L}$—a moment and a force respectively—should be applied to reach that $\boldsymbol{v}=\boldsymbol{0}$.

Choosing $\boldsymbol{L}$ time-dependent according to

$$\tau \frac{d\boldsymbol{L}}{dt} = -\boldsymbol{v}, \quad t>0 \tag{22}$$

and assuming a stable equilibrium of the bar at each time, it can easily be proved that the prescription (22) leads to $\boldsymbol{v}=\boldsymbol{0}$. The stable equilibrium demands with respect to $d\boldsymbol{L}\cdot d\boldsymbol{L}$ positive and finite work done by $d\boldsymbol{L}$, i.e.

$$\frac{d\boldsymbol{L}\cdot d\boldsymbol{v}}{d\boldsymbol{L}\cdot d\boldsymbol{L}} \geqq k>0. \tag{23}$$

Substituting (22) into (23) gives

$$\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} \leqq -\frac{k}{\tau}\boldsymbol{v}\cdot\boldsymbol{v} \tag{24}$$

or

$$\frac{d(\boldsymbol{v}\cdot\boldsymbol{v})}{dt}=2\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}\leqq-\frac{2k}{\tau}(\boldsymbol{v}\cdot\boldsymbol{v})\,. \tag{25}$$

Apparently $(\boldsymbol{v}\cdot\boldsymbol{v})$ diminishes not slower than in case of equality in (25). Hence

$$0\leqq(\boldsymbol{v}\cdot\boldsymbol{v})\leqq(\boldsymbol{v}\cdot\boldsymbol{v})_{t=0}\exp\left(-\frac{2k}{\tau}t\right) \tag{26}$$

i.e. the set of differential equations (22) is stable and

$$\boldsymbol{L}_{t=\infty}=\boldsymbol{L}_{\boldsymbol{v}=0}\,. \tag{27}$$

5. Analogue Simulation

Figures 3b,c show how the analogue simulation has been realized.

For the iteration-process over a cross-section during each iteration-cycle first the integration with respect to z is performed by means of the z-integrators. Then with the results of this integration the last estimates $\{u/R\}_n$ and $\{1/R\}_n$ are improved using the error-integrators, which integrate the differences N–N_n resp. M–M_n during a short time-interval T_c, resulting into similar equations as (11) and (12):

$$\left\{\frac{u}{R}\right\}_{n+1}=\left\{\frac{u}{R}\right\}_n+\frac{T_c}{\tau_N}(N-N_n) \tag{28}$$

$$\left\{\frac{1}{R}\right\}_{n+1}=\left\{\frac{1}{R}\right\}_n+\frac{T_c}{\tau_M}(M-M_n)\,. \tag{29}$$

The iteration-process over one cross-section has to be stopped after a finite number of iterations. Moreover only for a finite number (S) of cross-sections $1/R$ can be determined or in other words x has to be discretized as to the determination of $1/R$. Because succeeding cross-sections are very close together and therefore the corresponding solutions $1/R$ differ only slightly, in practice only one iteration per cross-section is made.

The integration with respect to x could be started as integration to time by the x-integrators as soon as the iteration-process over a cross-section has been finished until the next cross-section is reached. However assuming that this iteration-process is finished quasi-instantaneously, it is also possible to perform both integrations, to z as well to x, in parallel with respect to time, provided $x(t)$ is quasi-constant in comparison with $z(t)$.

In the implementation of the level-control iteration process for convenience sake the four time-intervals for forward and backward x-integration, for interpolating the new state and for updating the state are all taken of equal length. The interpolator consists of an integrator being in Compute-mode during the interpolating action. The level-control itself is realized by a level-integrator being in Compute-mode during the forward and backward x-integration. During the interpolation and the updating time-intervals the cross-section iteration-process is performed all the time over the cross-section for $x=0$. The level-control iteration-process continues during the whole computation.

The dynamic positioning of the simulated bar can be realized by continuous integration with respect to time of the differential equations

$$\frac{dM_3}{dt}=\theta_3-\frac{dw(x_3)}{dx}\,,\quad\frac{dQ_4}{dt}=w_4-w(x_4) \tag{30}$$

by means of integrators—which we will call positioners—provided that the data in the right-hand members are available quasi-continuously. These input-data of the positioners are renewed as soon as during the occurring level-control iteration-cycle the samples $dw(x_3)dx$ and $w(x_4)$ become available at the outputs of the corresponding track-stores. It means that the x-integration has to be performed quasi-instantaneously in comparison with the dynamic positioning.

Above as a degree of freedom in the design of the computation nesting of computing processes in different time-scales has been applied. This is technically possible for modern analogue and hybrid computers. This way of nesting of subroutines is very attractive. It allows the designer of the total computational scheme to focus first his attention to the specific parts of the computation and after that he can combine the resulting computing processes by superposition in different time-scales. It is not to be expected that this superposition will cause troubles because there is no strong dynamic interference between the different computing processes.

For instance if it was not allowed to design the dynamic positioning of the bar slowly compared to the level-control iteration-process, then it would be much more difficult to combine both computing processes. The designer would have to fight against the bad chance that both the stability of the dynamic positioning and the convergence of the level-control iteration-process are blown up because of the occurring dynamic interference.

6. Accuracy

Theoretically the final solution of a problem calculated according to the block-diagram of figure 3 is symmetric with respect to $x=l$ in each level-control iteration-cycle. This fact opens the possibility to get an idea by visual inspection of the quantitative effect of those sources of error which cause an asymmetry in the solution with respect to $x=l$.

Such a source of error is the time-delay which occurs in the determination of $1/R$ in forward as well as in backward x-integration as can easily be understood from the timing-scheme in figure 3c. This time-delay will increase by diminishing S, the number of cross-sections for which $1/R$ is determined.

Another source of error resulting into an asymmetry is the level-control. The level-integration will not only influence the level but also, however (mostly) slightly, the solution itself. This error will increase by diminishing the time-constants of the level-integrators. We remark that the influence on the solution will grow when the parameters of the bar come closer to a situation of buckling (the level-control iteration-process however remains convergent).

An error giving also an asymmetry is the small amount of damping if this is introduced in the x-integrators in order to improve the convergence of the level-control iteration-process; see [3]. In our experiments there was no reason to introduce such damping.

It will be clear that the total accuracy of the computation will depend on the characteristics of the available computing equipment but also on the parameters of the considered bar and on the computing parameters (for instance the value of S). A check showed that for the computing facilities used in this study (a home-built modern but low accurate analogue computer, 0.5% computing components) the computing accuracy was better than the accuracy of the oscilloscope display ($\simeq 2\%$) used as output-device.

7. Some Results

Figures 4, ..., 8 show some results in the form of oscilloscope pictures for a few examples.

Figure 4 represents the result of two successive integrations into the z-direction for a bar having a cross-section and a loading as indicated.

Figure 5 shows the deflection curves of a bar in two cases: a linear and a two-sided limited stress-strain relation.

The effect of the dynamic positioning in the case that the deflection in the middle has to be zero for this bar can be seen in figure 6. Figure 7 shows the transients in the outputs of the both level-integrators, the interpolator $dw(0)/dx$ and the positioner after starting the dynamic positioning.

Finally in figure 8 some results are given about buckling of a bar (loaded by a compressive force N) for the shown linear and nonlinear stress-strain relation and $w(0)=w(l)\neq 0$. In the nonlinear case the bar is weaker and as a consequence the first eigenvalue is smaller. Solutions are shown for N just smaller than the critical compressive force N_c and for N just larger than N_c. From the asymmetry of two of these solutions it can be concluded that the values of N in

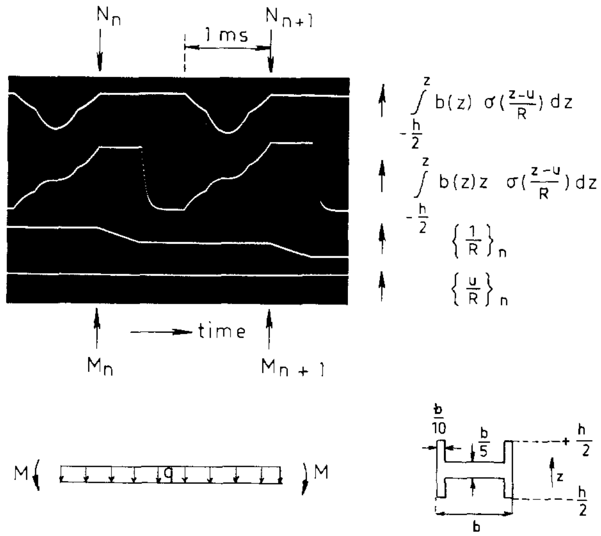


Figure 4. Two successive integrations into the z-direction.

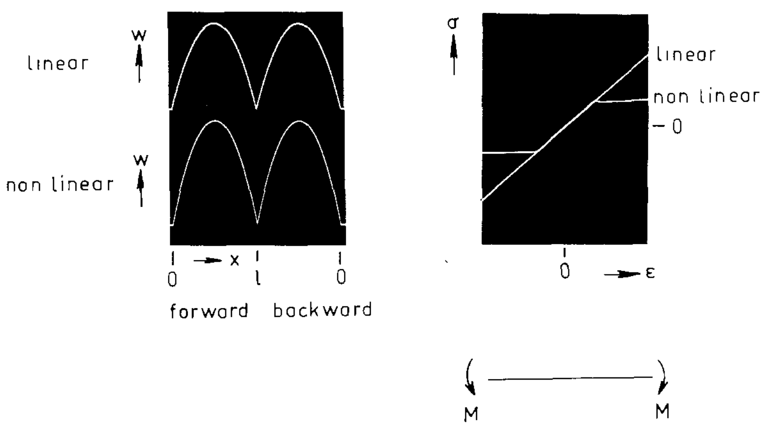
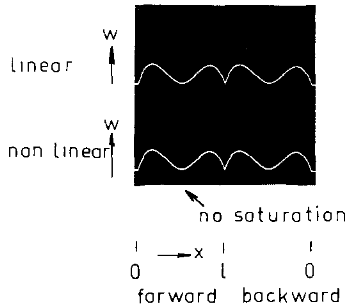


Figure 5. Bending of a linear and of a nonlinear bar.



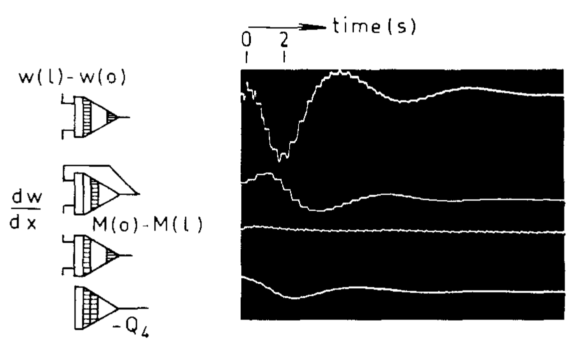


Figure 6. Dynamic positioning of the bar of Figure 5.
Figure 7. Transients after starting the dynamic positioning of the bar of Figure 5.

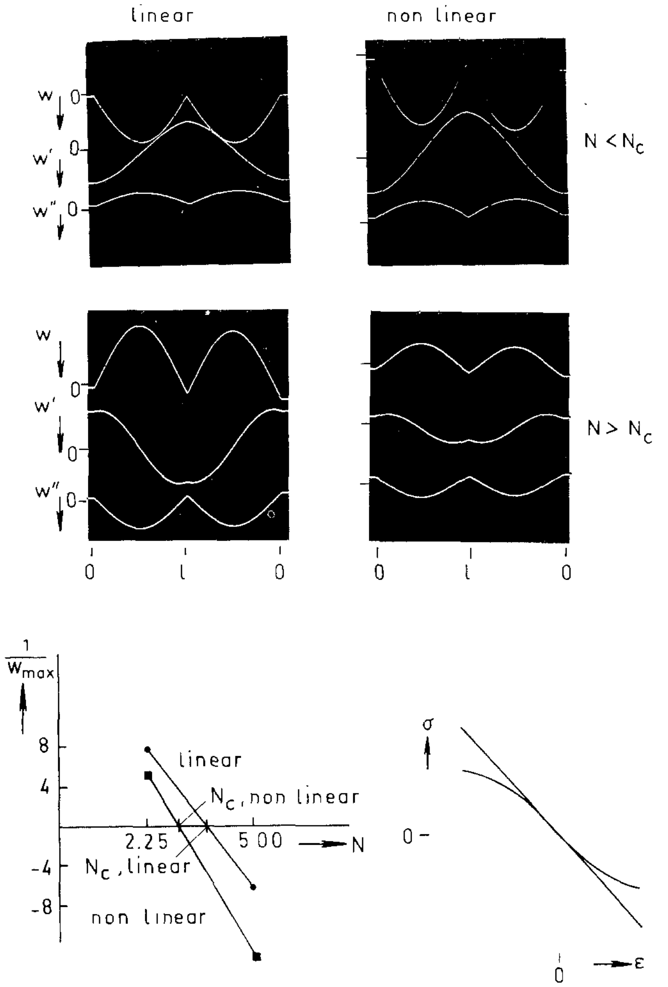


Figure 8. Buckling of a bar for a linear and a nonlinear stress-strain relation.

these cases are too near resonance; the level-control does influence the solutions noticeably. From linear interpolation it follows $N_{c\,\mathrm{linear}}/N_{c\,\mathrm{nonlinear}} \simeq 1{,}23$.

8. Final Remarks

The described computing method for the bending and buckling of a nonlinear elastic bar can be extended to more complicated problems: structure of bars, a vibrating bar, or a nonelastic bar.

A structure of bars can be handled with the described computing method without any difficulties. There are several possibilities. One can simulate each bar of the structure in the described way, where the boundary conditions for most of the bars are implicitly given, dependent on the way in which the bars are interconnected. These implicit boundary conditions can be dealt with in a similar way as the described dynamic positioning. In most structures many bars are lying in a straight line (i.e. horizontal and vertical lines). Then one can handle each straight line as one bar having for example implicitly given deflections as well as slopes in the nodes on this straight line. In each node the implicit conditions are coupled for all straight lines connected to this node. The first mentioned possibility asks more computing equipment, the second one more computing time.

The extension of the problem to vibrating bars asks for storing of (two) previous states of the bar if the time is discretized. Having available a hybrid computer the vibration problem in this form can be attacked straightforward by using the digital computer as a storing device. Using only analogue means one can consider the bar as a jumped-mass system as to the vibrating aspect. Then the bending of the bar can be simulated in the known way and the vibration is described by a finite number of second order differential equations to time for these masses. These differential equations can be solved by means of integrators provided the chosen time-scale is slow enough to allow again the superposition of all the different computing processes.

In our investigation the presence of elasticity or in other words of a unique relation between σ and ε has been an essential assumption. The nonelastic case is much more difficult to deal with. As soon as the yield point has been reached in some point (x, z) of the bar it is no longer possible to determine new values (σ, ε) without having available the previous values of (σ, ε). It means that it is necessary to store each solution in order to be able to compute the next solution for a slightly changed loading. This problem cannot be solved with merely analogue means, but asks for a hybrid computer [4].

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